

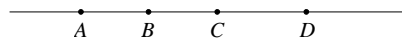
# Harmonic Division and its Applications

Cosmin Pohoata

Let  $d$  be a line and  $A, B, C$  and  $D$  four points which lie in this order on it. The four-point  $(ABCD)$  is called a *harmonic division*, or simply *harmonic*, if (using directed lengths)

$$\frac{CA}{CB} = -\frac{DA}{DB}.$$

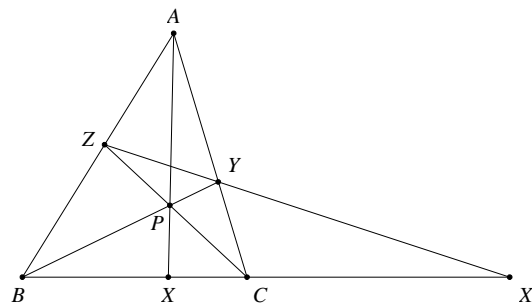
If  $X$  is a point not lying on  $d$ , then we say that pencil  $X(ABCD)$  (which consists of the four lines  $XA, XB, XC, XD$ ) is harmonic if  $(ABCD)$  is harmonic.



In this note, we show how to use harmonic division as a tool in solving some difficult Euclidean geometry problems.

We begin by stating two very useful lemmas without proof. The first lemma shows one of the simplest geometric characterizations of harmonic divisions, based on the theorems of Menelaus and Ceva.

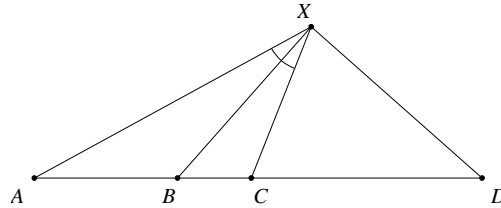
**Lemma 1.** *In a triangle  $ABC$  consider three points  $X, Y, Z$  on the sides  $BC, CA$ , respective  $AB$ . If  $X'$  is the point of intersection of  $YZ$  with the extended side  $BC$ , then the four-point  $(BXCX')$  forms and harmonic division if and only if the cevians  $AX, BY$  and  $CZ$  are concurrent.*



The second lemma is a consequence of the Apollonius circle property. It can be found in [1] followed by several interpretations.

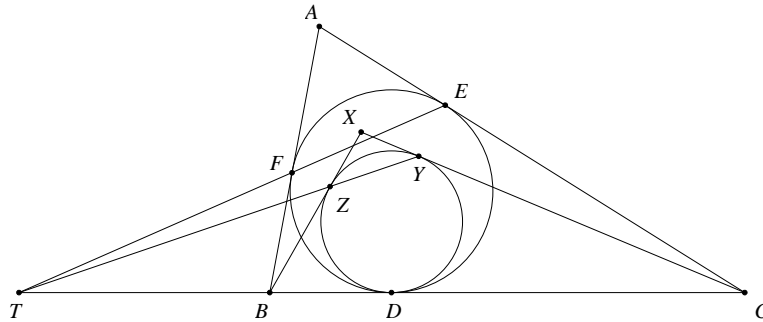
**Lemma 2.** *Let four points  $A, B, C$  and  $D$ , in this order, lying on  $d$ . Then, if two of the following three propositions are true, then the third is also true:*

- (1) The division  $(ABCD)$  is harmonic.
- (2)  $XB$  is the internal angle bisector of  $\angle AXC$ .
- (3)  $XB \perp XD$ .



We begin our journey with a problem from the IMO 1995 Shortlist.

**Problem 1.** Let  $ABC$  be a triangle, and let  $D, E, F$  be the points of tangency of the incircle of triangle  $ABC$  with the sides  $BC, CA$  and  $AB$  respectively. Let  $X$  be in the interior of  $ABC$  such that the incircle of  $XBC$  touches  $XB, XC$  and  $BC$  in  $Z, Y$  and  $D$  respectively. Prove that  $EFZY$  is cyclic.

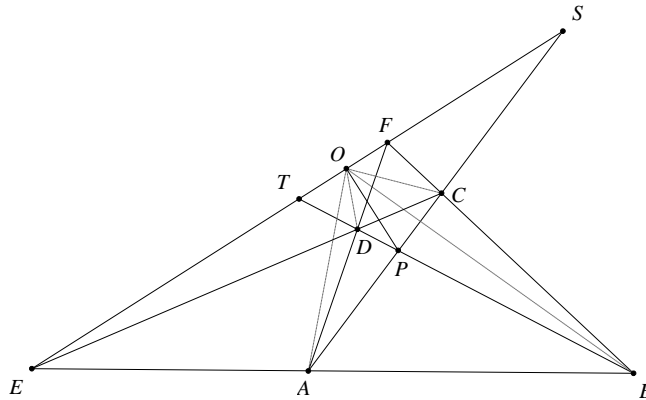


*Solution.* Denote  $T = BC \cap EF$ . Because of the concurrency of the lines  $AD, BE, CF$  in the Gergonne point of triangle  $ABC$ , we deduce that the division  $(TBDC)$  is harmonic. Similarly, the lines  $XD, BY$  and  $CZ$  are concurrent in the Gergonne point of triangle  $XBC$ , so  $T \in YZ$  as a consequence of Lemma 1.

Now expressing the power of point  $T$  with respect to the incircle of triangle  $ABC$  and the incircle of triangle  $XBC$  we have that  $TD^2 = TE \cdot TF$  and  $TD^2 = TZ \cdot TY$ . So  $TE \cdot TF = TZ \cdot TY$ , therefore the quadrilateral  $EFZY$  is cyclic.  $\square$

For our next application, we present a problem given at the Chinese IMO Team Selection Test in 2002.

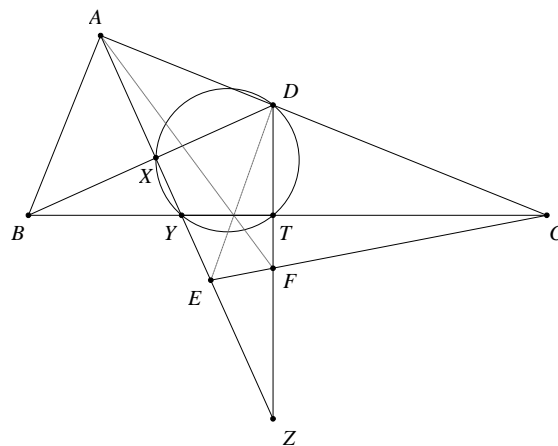
**Problem 2.** Let  $ABCD$  be a convex quadrilateral. Let  $E = AB \cap CD, F = AD \cap BC, P = AC \cap BD$ , and let  $O$  the foot of the perpendicular from  $P$  to the line  $EF$ . Prove that  $\angle BOC = \angle AOD$ .



*Solution.* Denote  $S = AC \cap EF$  and  $T = BD \cap EF$ . As from Lemma 1, we deduce that the division  $(ETFS)$  is harmonic. Furthermore, the division  $(APCS)$  is also harmonic, due to the pencil  $B(ETFS)$ . But now, the pencil  $E(APCS)$  is harmonic, so by intersecting it with the line  $BD$ , it follows that the four-point  $(BPDT)$  is harmonic. Therefore, the pencil  $O(APCS)$  is harmonic and  $OP \perp OS$ , thus by Lemma 2,  $\angle POA = \angle POC$ . Similarly, the pencil  $O(BPDT)$  is harmonic and  $OP \perp OT$ , thus again by Lemma 2,  $\angle POB = \angle POD$ . It follows that  $\angle AOD = \angle BOC$ .  $\square$

We continue with an interesting problem proposed by Dinu Serbanescu at the Romanian Junior Balkan MO 2007, Team Selection Test.

**Problem 3.** Let  $ABC$  be a right triangle with  $\angle A = 90^\circ$  and let  $D$  be a point on side  $AC$ . Denote by  $E$  the reflection of  $A$  across the line  $BD$  and  $F$  the intersection point of  $CE$  with the perpendicular to  $BC$  at  $D$ . Prove that  $AF$ ,  $DE$  and  $BC$  are concurrent.



*Solution.* Denote the points  $X = AE \cap BD$ ,  $Y = AE \cap BC$ ,  $Z = AE \cap DF$  and  $T = DF \cap BC$ . From Lemma 1, applied to triangle  $AEC$  and for the cevians  $AF$

and  $ED$ , we observe that the lines  $AF$ ,  $DE$  and  $BC$  are concurrent if and only if the division  $(AYEZ)$  is harmonic.

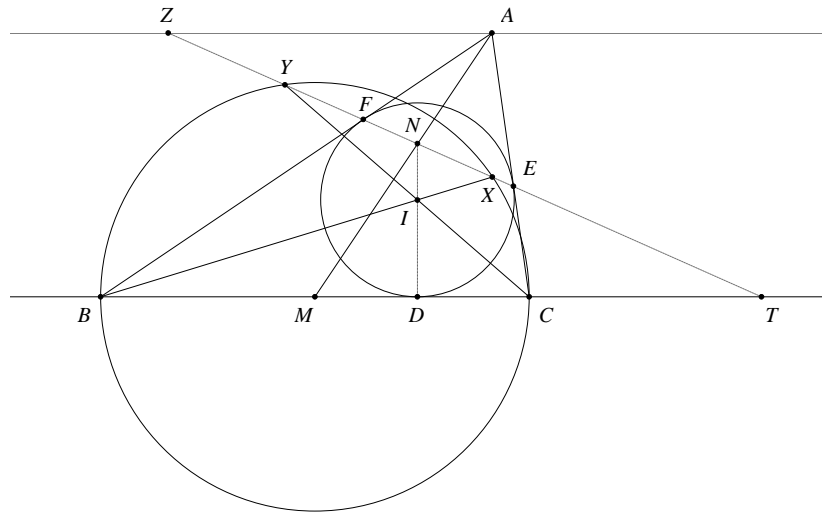
Since the quadrilateral  $XYTD$  is cyclic,  $\tan XYB = \tan XDZ$ , which is equivalent to  $XB/XY = XZ/XD$ . So  $XB \cdot XD = XY \cdot XZ$ .

Since triangles  $XAB$  and  $XDA$  are similar, we have that  $XA^2 = XB \cdot XD$ , so  $XA^2 = XY \cdot XZ$ . Using  $XA = XE$ , we obtain that  $\frac{YA}{YE} = \frac{ZA}{ZE}$ , and thus the division  $(AYEZ)$  is harmonic.  $\square$

The next problem was proposed by the author and given at the Romanian IMO Team Selection Test in 2007.

**Problem 4.** Let  $ABC$  be a triangle, let  $E, F$  be the tangency points of the incircle  $\Gamma(I)$  to the sides  $AC$ , respectively  $AB$ , and let  $M$  be the midpoint of the side  $BC$ . Let  $N = AM \cap EF$ , let  $\gamma(M)$  be the circle of diameter  $BC$ , and let lines  $BI$  and  $CI$  meet  $\gamma$  again at  $X$  and  $Y$ , respectively. Prove that

$$\frac{NX}{NY} = \frac{AC}{AB}.$$



*Solution.* We will assume  $AB \geq AC$ , so the solution matches a possible drawing. Let  $T = EF \cap BC$  (for  $AB = AC$ ,  $T = \infty$ ), and  $D$  the tangency point of  $\Gamma$  to  $BC$ .

**Claim 1.** In the configuration described above, for  $X' = BI \cap EF$ , one has  $BX' \perp CX'$ .

*Proof.* The fact that  $BI$  effectively intersects  $EF$  follows from  $\angle DFE = \frac{1}{2}(\angle ABC + \angle BAC) = \frac{1}{2}\pi - \frac{1}{2}\angle ACB < \frac{1}{2}\pi$ , and  $BI \perp DF$  (similarly,  $CI$  effectively intersects  $EF$ ).

The division  $(TBDC)$  is harmonic, and triangles  $BFX'$  and  $BDX'$  are congruent, therefore  $\angle TX'B = \angle DX'B$ , which is equivalent to  $BX' \perp CX'$  (similarly, for  $Y' = CI \cap EF$ , one has  $CY' \perp BY'$ ).  $\square$

**Claim 2.** In the configuration described above, one has  $N = DI \cap EF$ .

*Proof.* It is enough to prove that  $NI \perp BC$ . Let  $d$  be the line through  $A$ , parallel to  $BC$ . Since the pencil  $A(BMC\infty)$  is harmonic, it follows the division  $(FNEZ)$  is harmonic, where  $Z = d \cap EF$ . Therefore  $N$  lies on the polar of  $Z$  relative to circle  $\Gamma$ , and as  $N \in EF$  (the polar of  $A$ ), it follows that  $AZ$  is the polar of  $N$  relative to circle  $\Gamma$ , hence  $NI \perp d$ , so  $NI \perp BC$ . In conclusion, since  $DI \perp BC$ , one has  $N \in DI$ .  $\square$

It follows, according to Claim 1, that  $X = X'$  and  $Y = Y'$ , therefore  $X, Y \in EF$ . Since the division  $(TBDC)$  is harmonic, it follows that  $D$  lies on the polar  $p$  of  $T$  relative to circle  $\gamma$ . But  $TM \perp p$ , so  $BC \perp p$ , and since  $DI \perp BC$ , it follows that  $p$  is, in fact,  $DI$ .

Now, according to Claim 2, it follows that  $D, I, N$  are collinear. Since  $DN$  is the polar, it means the division  $(TYNX)$  is harmonic, thus the pencil  $D(TYNX)$  is harmonic. But  $DT \perp DN$ , so  $DN$  is the angle bisector of  $\angle XDY$ , hence

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin \angle DYX}{\sin \angle DXY}.$$

As quadrilaterals  $BDIY$  and  $CDIX$  are cyclic (since pairs of opposing angles are right angles), it follows that  $\frac{1}{2}\angle ABC = \angle DBI = \angle DYI = \frac{1}{2}\angle DYX$  (triangles  $CDY$  and  $CEY$  are congruent), so  $\angle DYX = \angle ABC$ . Similarly,  $\angle DXY = \angle ACB$ . Therefore

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin \angle DYX}{\sin \angle DXY} = \frac{\sin \angle ABC}{\sin \angle ACB} = \frac{AC}{AB}.$$

$\square$

The following problem was posted on the MathLinks forum [2]:

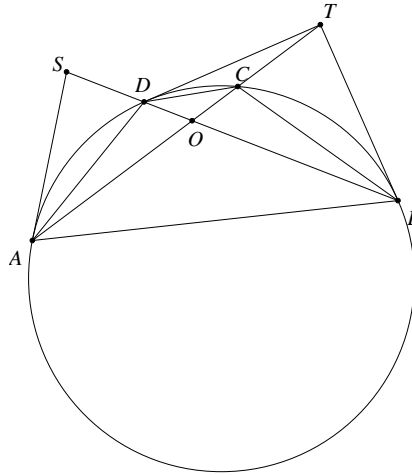
**Problem 5.** Let  $ABC$  be a triangle and  $\rho(I)$  its incircle.  $D, E$  and  $F$  are the points of tangency of  $\rho(I)$  with  $BC, CA$  and  $AB$  respectively. Denote  $M = \rho(I) \cap AD$ ,  $N$  the intersection of the circumcircle of  $CDM$  with  $DF$  and  $G = CN \cap AB$ . Prove that  $CD = 3FG$ .



Let  $ABCD$  be a cyclic quadrilateral and  $X$  a point on the circle. Then, the  $ABCD$  is called *harmonic* if the pencil  $X(ABCD)$  is harmonic. For a list of properties regarding the harmonic quadrilateral, interested readers may consult [1] and [3].

The following problem was given at an IMO Team Preparation Contest, held in Bacau, Romania, in 2006.

**Problem 6.** Let  $ABCD$  be a convex quadrilateral, for which denote  $O = AC \cap BD$ . If  $BO$  is a symmedian of triangle  $ABC$  and  $DO$  is a symmedian of triangle  $ADC$ , prove that  $AO$  is a symmedian of triangle  $ABD$ .



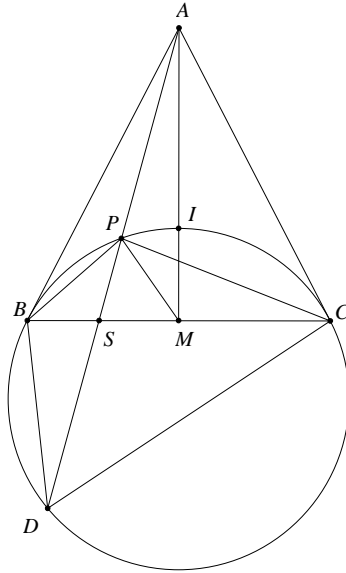
*Solution.* Denote  $T_1 = DD \cap AC$ ,  $T_2 = BB \cap AC$ ,  $T = BB \cap DD$ , where  $DD$ , respective  $BB$  represents the tangent in  $D$  to the circumcircle of  $ADC$  and the tangent in  $B$  to the circumcircle of  $ABC$ .

Since  $BO$  is a symmedian of triangle  $ABC$  and  $DO$  is a symmedian of triangle  $ADC$ , the divisions  $(AOCT_1)$  and  $(AOCT_2)$  are harmonic, so  $T_1 = T_2 = T$ .

Hence,  $BD$  is the polar of  $T_1$  with respect to the circumcircle of  $ADC$  and also the polar of  $T_2$  with respect to the circumcircle of  $ABC$ . But because  $T_1 = T_2$ , we deduce that the circles  $ABC$  and  $ADC$  coincide, i.e. the quadrilateral  $ABCD$  is cyclic, and since the division  $(AOCT)$  is harmonic, the pencil  $D(AOCT)$  is, and by intersecting it by the circle  $ABCD$ , it follows that the quadrilateral  $ABCD$  is also harmonic. Then, the pencil  $A(ABCD)$  is harmonic. By intersecting it with the line  $BD$ , we see that the division  $(BODS)$  is harmonic, where  $S = AA \cap BD$ . It follows that  $AO$  is a symmedian of triangle  $BAD$ .  $\square$

The next problem was also given in an IMO Team Preparation Test, at the IMAR Contest, held in Bucharest in 2006.

**Problem 7.** Let  $ABC$  be an isosceles triangle with  $AB = AC$ , and  $M$  the midpoint of  $BC$ . Find the locus of the point  $P$  interior to the triangle for which  $\angle BPM + \angle CPA = \pi$ .



*Solution.* Denote the point  $D$  as the intersection of the line  $AP$  with the circumcircle of  $BPC$  and  $S = DP \cap BC$ .

Since  $\angle SPC = 180 - \angle CPA$ , it follows that  $\angle BPS = \angle CPM$ .

From the Steiner theorem applied in to triangle  $BPC$  for the isogonals  $PS$  and  $PM$ ,

$$\frac{SB}{SC} = \frac{PB^2}{PC^2}.$$

On other hand, using Sine Law, we obtain

$$\frac{SB}{SC} = \frac{DB}{DC} \cdot \frac{\sin \angle SDB}{\sin \angle SDC} = \frac{DB}{DC} \cdot \frac{\sin \angle PCB}{\sin \angle PBC} = \frac{DB}{DC} \cdot \frac{PB}{PC}.$$

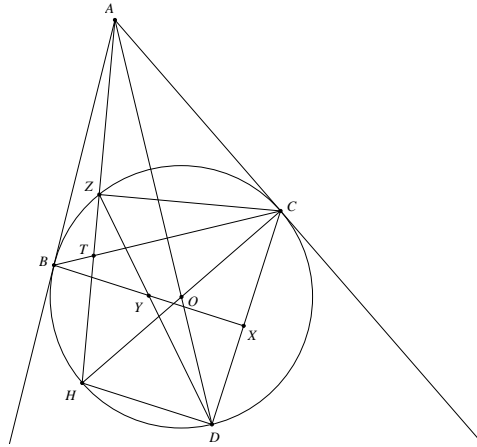
Thus by the above relations, it follows that  $\frac{DB}{DC} = \frac{PB}{PC}$ , i.e. the quadrilateral  $PBDC$  is harmonic, therefore the point  $A' = BB \cap CC$  lies on the line  $PD$ .

If  $A' = A$ , then lines  $AB$  and  $AC$  are always tangent to the circle  $BPC$ , and so the locus of  $P$  is the circle  $BIC$ , where  $I$  is the incircle of  $ABC$ . Otherwise, if  $A' \neq A$ , then  $A' = AM \cap PS \cap BB \cap CC$ , due to the fact that  $A' \in PD$  and  $A = PS \cap AM$ , therefore by maintaining the condition that  $A' \neq A$ , we obtain that  $PS = AM$ , therefore  $P$  lies on  $AM$ .  $\square$

The next problem was selected in the Senior BMO 2007 Shortlist, proposed by the author.

**Problem 8.** Let  $\rho(O)$  be a circle and  $A$  a point outside it. Denote by  $B, C$  the points where the tangents from  $A$  with respect to  $\rho(O)$  meet the circle,  $D$  the point on  $\rho(O)$ , for which  $O \in AD$ ,  $X$  the foot of the perpendicular from  $B$  to  $CD$ ,  $Y$  the midpoint of the line segment  $BX$  and by  $Z$  the second intersection of  $DY$  with  $\rho(O)$ . Prove that  $ZA \perp ZC$ .



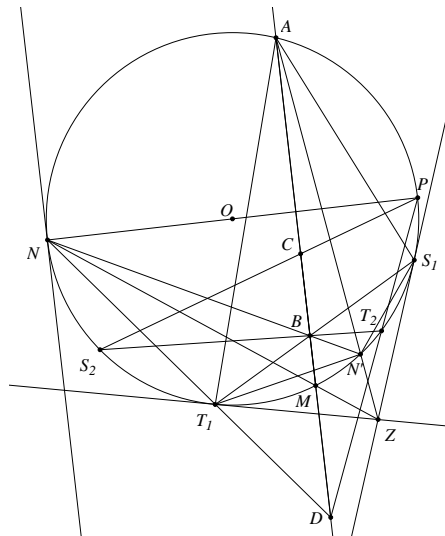


*Solution.* Let us call  $H = CO \cap \rho(O)$ . Thus  $DC \perp DH$ , so  $DH \parallel BX$ .

Because  $Y$  is the midpoint of  $BX$ , we deduce that the division  $(BYX\infty)$  is harmonic, so also is the pencil  $D(BYXH)$  and by intersecting it with  $\rho(O)$ , it follows that the quadrilateral  $HBZC$  is harmonic. Then, the pencil  $C(HBZC)$  is harmonic, so by intersecting it with the line  $HZ$ , it follows that the division  $(A'ZTH)$  is harmonic, where  $A' = HZ \cap CC$  and  $T = HZ \cap BC$ .

So, the line  $CH$  is the polar of  $A'$  with respect to  $\rho(O)$ , but  $CH = BC$  is the polar of  $A$  as well, so  $A = A'$ , hence the points  $H, Z, A$  are collinear, therefore  $ZA \perp ZC$ .  $\square$

The last problem is a generalization of a problem by Virgil Nicula [4]. The solution covers all concepts and methods presented throughout this paper.



**Problem 9.** Let  $d$  be a line and  $A, C, B, D$  four points in this order on it such that the division  $(ACBD)$  is harmonic. Denote by  $M$  the midpoint of the line segment  $CD$ . Let  $\omega$  be a circle passing through  $A$  and  $M$ . Let  $NP$  be the diameter of  $\omega$  perpendicular to  $AM$ . Let lines  $NC, ND, PC, PD$  meet  $\omega$  again at  $S_1, T_1, S_2, T_2$ , respectively. Prove that  $B = S_1T_1 \cap S_2T_2$ .

*Solution.* Since the four-point  $(ACBD)$  is harmonic, so is the pencil  $N(ACBD)$  and by intersecting it with  $\omega$ , it follows that the quadrilateral  $AS_1N'T_1$  is harmonic, hence the lines  $S_1S_1, T_1T_1$  and  $AN'$  are concurrent, where  $N' = NB \cap \omega$ .

Because the tangent in  $N$  to  $\omega$  is parallel with the line  $AM$  and since  $M$  is the midpoint of  $CD$ , the division  $(CMD\infty)$  is harmonic, therefore the pencil  $N(NDMC)$  also is, and by intersecting it with  $\omega$ , it follows that the quadrilateral  $NT_1MS_1$  is harmonic, hence the lines  $S_1S_1, T_1T_1$  and  $MN$  are concurrent.

From the above two observations, we deduce that the lines  $S_1S_1, T_1T_1, MN, AN'$  are concurrent at a point  $Z$ .

On the other hand, since the pencils  $B(AS_1N'T_1)$  and  $B(NT_1MS_1)$  are harmonic, by intersecting them with  $\omega$ , it follows that the quadrilaterals  $NT_3MS_3$  and  $AS_3N'T_3$  harmonic, where  $S_3 = BS_1 \cap \omega$  and  $T_3 = BT_1 \cap \omega$ .

Similarly, we deduce that the lines  $S_3S_3, T_3T_3, MN$  and  $AN'$  are concurrent in the same point  $Z$ .

Therefore,  $S_3T_3$  is the polar of  $Z$  with respect to  $\omega$ , but so is  $S_1T_1$ , thus  $S_1T_1 = S_3T_3$ , so  $S_1 = S_3$  and  $T_1 = T_3$ , therefore the points  $S_1, B, T_1$  are collinear.

Similarly, the points  $S_2, B, T_2$  are collinear, from which it follows that  $B = S_1T_1 \cap S_2T_2$ .  $\square$

## References

- [1] Virgil Nicula, Cosmin Pohoata. *Diviziunea armonica*. GIL, 2007.
- [2] <http://www.mathlinks.ro/viewtopic.php?t=151320>
- [3] <http://www.mathlinks.ro/viewtopic.php?t=70184>
- [4] <http://www.mathlinks.ro/viewtopic.php?t=155875>

Cosmin Pohoata  
pohoata\_cosmin2000@yahoo.com